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Research Article

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Long-Scale Ollivier Ricci Curvature of Graphs

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Abstract: We study the long-scale Ollivier Ricci curvature of graphs as a function of the chosen idleness. Similarly to the previous work on the short-scale case, we show that this idleness function is concave and piecewise linear with at most 3 linear parts. We provide bounds on the length of the first and last linear pieces. We also study the long-scale curvature for the Cartesian product of two regular graphs.

Keywords: curvature; graphs; optimal transport

MSC: 05C10, 52C99, 90C08

1 Introduction and statement of results

Ricci curvature is a fundamental notion in the study of Riemannian manifolds. This notion has been generalized in various ways from the smooth setting of manifolds to more general metric spaces. For example, in [11] Ollivier introduced a notion of Ricci curvature on metric spaces (later known as “Ollivier Ricci curvature”). This gives rise to a notion of Ricci curvature on graphs taking values on the edges and based on optimal transport of lazy random walks, with respect to an idleness parameter p . In [6] this notion was modified on graphs to give the “Lin-Lu-Yau” curvature.

Beyond recent theoretical work on this notion (see [1–4, 7, 12]), there have been several applications outside mathematics such as in biology (see [5, 13, 17]) and in computing (see [10, 15, 16]).

In [1] the authors investigate the Ollivier Ricci idleness function $p \mapsto \kappa_p(x, y)$, which takes the idleness parameter $p \in [0, 1]$ as a variable and gives the value of curvature between the fixed two adjacent vertices x and y (or equivalently, the curvature given on an edge of the graph joining x and y). End this sentence with a full-stop, after the intervals expression $p \mapsto \kappa_p(x, y)$ is concave and piecewise linear on $[0, 1]$ with at most 3 linear parts, and it is linear on the intervals

$$\left[0, \frac{1}{\text{lcm}(d_x, d_y) + 1}\right] \text{ and } \left[\frac{1}{\max(d_x, d_y) + 1}, 1\right]$$

In this paper, we do similar investigation on the idleness function, but the condition that the two vertices are adjacent is replaced by distance ≥ 2 apart (and henceforth called “long-scale curvature” as in contrast to “short-scale curvature”). Our main result is that the (long-scale) idleness function $p \mapsto \kappa_p(x, y)$ is concave and piecewise linear on $[0, 1]$ with at most 3 linear parts, and it is linear on the intervals

$$\left[0, \frac{1}{\text{lcm}(d_x, d_y) + 1}\right] \text{ and } \left[\frac{1}{2} - \frac{1}{2} \cdot \frac{d_x + d_y}{2d_x d_y - d_x - d_y}, 1\right].$$

In a specific case when $d_x = 1$ or $d_y = 1$, the idleness function $p \mapsto \kappa_p(x, y)$ is linear on the entire interval $[0, 1]$.

This main result is split into two theorems, which are stated and proved in Section 3 and 4. In Section 5, we provide an example of a graph that has exactly 3 linear parts and the first and the last linear parts

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are the same intervals as mentioned in the main result. In Section 6, we give the formula of the long-scale curvature of Cartesian products of regular graphs. In Section 7, we present some interesting behaviours of the long-scale curvature, including the hexagonal tiling, and the discrete Bonnet-Myers' theorem.

2 Definitions and notation

We now introduce the relevant definitions and notation we will need in this paper.

Throughout this article, let $G = (V, E)$ be a simple graph (i.e., G contains no multiple edges or self loops) with a vertex set V and an edge set E . Furthermore, we assume that G is locally finite and connected. Let $d_x \in \mathbb{N}$ denote the degree of the vertex $x \in V$ and $d(x, y) \in \mathbb{N} \cup \{0\}$ denote the combinatorial distance, that is, the length of a shortest path (also called a geodesic) between two vertices x and y . We also denote the existence of an edge between x and y by $x \sim y$.

A *probability measure* μ on V is a function $\mu : V \rightarrow [0, 1]$ satisfying $\sum_{v \in V} \mu(v) = 1$. All probability measures are assumed to be finitely supported, that is

$$\text{supp}(\mu) = \{v \in V \mid \mu(v) > 0\}$$

is a finite set. For any $x \in V$ and $p \in [0, 1]$, the probability measure μ_x^p is defined as

$$\mu_x^p(v) = \begin{cases} p & \text{if } v = x \\ \frac{1-p}{d_x} & \text{if } v \sim x \\ 0 & \text{otherwise.} \end{cases}$$

A 1-sphere and a 1-ball around a vertex $x \in V$ are defined as

$$\begin{aligned} S_1(x) &= \{v \in V : v \sim x\} \\ B_1(x) &= \{v \in V : v = x \text{ or } v \sim x\} = S_1(x) \cup \{x\}. \end{aligned}$$

In particular, $\text{supp}(\mu_x^p) \subseteq B_1(x)$ for all $p \in [0, 1]$.

Let W_1 denote the 1-Wasserstein distance between two probability measures (see [14, pp. 211]). Its definition on graphs can be written as follows.

Definition 2.1. Let $G = (V, E)$ be a locally finite and connected graph. Let μ_1, μ_2 be two probability measures on V . The *Wasserstein distance* $W_1(\mu_1, \mu_2)$ between μ_1 and μ_2 is defined as

$$W_1(\mu_1, \mu_2) = \inf_{\pi \in \Pi(\mu_1, \mu_2)} \sum_{w \in V} \sum_{v \in V} d(v, w) \pi(v, w), \quad (2.1)$$

where

$$\Pi(\mu_1, \mu_2) = \left\{ \pi : V \times V \rightarrow [0, 1] : \mu_1(v) = \sum_{w \in V} \pi(v, w), \mu_2(w) = \sum_{v \in V} \pi(v, w) \right\}.$$

A function $\pi \in \Pi(\mu_1, \mu_2)$ is called a *transport plan* transporting μ_1 to μ_2 (and π is said to have marginals μ_1 and μ_2). In words, a transport plan π moves a mass distribution given by μ_1 into a mass distribution given by μ_2 , and $W_1(\mu_1, \mu_2)$ is a measure for the minimal effort which is required for such a transition. If π attains the infimum in (2.1) we call it an *optimal transport plan* transporting μ_1 to μ_2 .

Note that $\text{supp}(\pi) \subseteq \text{supp}(\mu_1) \times \text{supp}(\mu_2)$ for all $\pi \in \Pi(\mu_1, \mu_2)$. Consequently, the marginal constraints can be written as:

$$\mu_1(v) = \sum_{w \in \text{supp}(\mu_2)} \pi(v, w), \quad \mu_2(w) = \sum_{v \in \text{supp}(\mu_1)} \pi(v, w),$$

and the defining equation (2.1) can be written as

$$W_1(\mu_1, \mu_2) = \inf_{\pi \in \Pi(\mu_1, \mu_2)} \sum_{w \in \text{supp}(\mu_2)} \sum_{v \in \text{supp}(\mu_1)} d(v, w) \pi(v, w). \quad (2.2)$$

The Ollivier Ricci curvature is introduced in [11] and further modified by Lin, Lu, and Yau in [6] as follows.

Definition 2.2. Let $G = (V, E)$ be a locally finite and connected graph. For $p \in [0, 1]$, the p -Ollivier Ricci curvature of two different vertices $x, y \in V$ is

$$\kappa_p(x, y) = 1 - \frac{W_1(\mu_x^p, \mu_y^p)}{d(x, y)},$$

where $p \in [0, 1]$ is called the *idleness* parameter.

Moreover, the Lin-Lu-Yau curvature is

$$\kappa_{LLY}(x, y) = \lim_{p \rightarrow 1} \frac{\kappa_p(x, y)}{1 - p}.$$

In particular, we call the curvature $\kappa_p(x, y)$ and $\kappa_{LLY}(x, y)$ “short-scale” when $x \sim y$, and we call it “long-scale” when $d(x, y) \geq 2$.

A fundamental concept in optimal transport theory and vital to our work is Kantorovich duality. First we recall the notion of 1-Lipschitz functions and then state the Kantorovich duality theorem.

Definition 2.3. Let $G = (V, E)$ be a locally finite and connected graph. For any $U \subseteq V$, a function $\phi : U \rightarrow \mathbb{R}$ is 1-Lipschitz (on U) if for all $v, w \in U$,

$$|\phi(v) - \phi(w)| \leq d(v, w).$$

Furthermore, let 1-Lip denote the set of all 1-Lipschitz functions on V .

Theorem 2.1 (Kantorovich duality [14]). *Let $G = (V, E)$ be a locally finite and connected graph. Let μ_1, μ_2 be two probability measures on V . Then*

$$W_1(\mu_1, \mu_2) = \sup_{\substack{\phi: V \rightarrow \mathbb{R} \\ \phi \in 1\text{-Lip}}} \sum_{x \in V} \phi(x) (\mu_1(x) - \mu_2(x)). \quad (2.3)$$

If $\phi \in 1\text{-Lip}$ attains the supremum we call it an *optimal Kantorovich potential* transporting μ_1 to μ_2 .

Remark 2.2 (Existence of optimal transport plans and optimal Kantorovich potentials). The defining equation (2.2) can be realized as a finite-dimensional linear minimization problem on the bounded convex set $\Pi(\mu_1, \mu_2)$, so this problem admits a minimizer π . Furthermore, its dual problem can be written as

$$W_1(\mu_1, \mu_2) = \sup_{\substack{\Phi: U \rightarrow \mathbb{R} \\ 1\text{-Lipschitz on } U}} \sum_{x \in U} \Phi(x) (\mu_1(x) - \mu_2(x)),$$

where $U = \text{supp}(\mu_1) \cup \text{supp}(\mu_2) \subseteq V$.

It is well-known that any 1-Lipschitz function on $U \subseteq V$ can always be extended to a 1-Lipschitz function on V (see [8]). In particular, a maximizer Φ is extended to a maximizer ϕ in the equation (2.3). This argument guarantees the existence of an optimal transport plan and an optimal Kantorovich potential transporting μ_1 to μ_2 .

We now present the complementary slackness theorem in the setting of our optimal transport problem, which says that the 1-Lipschitz condition on any optimal Kantorovich potential holds with equality on the support of any optimal transport plan. A similar statement and a proof can be found in [1, Lemma 3.1].

Theorem 2.3 (The complementary slackness [1]). *Let $G = (V, E)$ be a locally finite and connected graph. Let μ_1, μ_2 be two probability measures on V , and let π and ϕ be an optimal transport plan and an optimal Kantorovich potential transporting μ_1 to μ_2 . Then for all $v, w \in V$ such that $\pi(v, w) > 0$,*

$$\phi(v) - \phi(w) = d(v, w).$$

Most of the time, it is sufficient to consider 1-Lipschitz functions that only yield integer values. This observation is an important tool to deal with Kantorovich potential as in the following way.

Definition 2.4. Let $G = (V, E)$ be a locally finite and connected graph and let $\phi : V \rightarrow \mathbb{R}$. Define two functions $\lfloor \phi \rfloor, \lceil \phi \rceil : V \rightarrow \mathbb{Z}$ to be $\lfloor \phi \rfloor(v) := \lfloor \phi(v) \rfloor$ and $\lceil \phi \rceil(v) := \lceil \phi(v) \rceil$.

Lemma 2.4. ([1, Lemma 3.2]) *Let $G = (V, E)$ be a locally finite graph. Suppose that a function $\phi : V \rightarrow \mathbb{R}$ is 1-Lipschitz. Then the functions $\lfloor \phi \rfloor$ and $\lceil \phi \rceil$ are also 1-Lipschitz.*

The existence of integer-valued optimal Kantorovich potentials can be formulated as in the following proposition, which generalizes the result from [1, Lemma 3.3].

Proposition 2.5 (Integer-Valuedness). *Let $G = (V, E)$ be a locally finite and connected graph. Let μ_1, μ_2 be two probability measures on V . Then there exists an integer-valued optimal Kantorovich potential $\phi : V \rightarrow \mathbb{Z}$ transporting μ_1 to μ_2 , that is*

$$W_1(\mu_1, \mu_2) = \sum_{x \in V} \phi(x)(\mu_1(x) - \mu_2(x)). \quad (2.4)$$

Proof. Let a 1-Lipschitz function $\phi^* : V \rightarrow \mathbb{R}$ be an optimal Kantorovich potential transporting μ_1 to μ_2 , that is

$$W_1(\mu_1, \mu_2) = \sum_{x \in V} \phi^*(x)(\mu_1(x) - \mu_2(x)).$$

By Proposition 2.4, the function $\lfloor \phi^* \rfloor \in 1\text{-Lip}$. We will show that $\phi = \lfloor \phi^* \rfloor$ satisfies (2.4), and it is therefore an integer-valued optimal Kantorovich potential.

For each $v \in V$, define $\delta_v = \phi^*(v) - \lfloor \phi^*(v) \rfloor \in [0, 1)$ to be the fractional part of $\phi^*(v)$. Let π be an optimal transport plan transporting μ_1 to μ_2 , and denote $U = \text{supp}(\mu_1) \cup \text{supp}(\mu_2)$. Construct a graph H with vertices in U and edges given by its adjacency matrix A_H :

$$A_H(v, w) = \begin{cases} 1 & \text{if } \pi(v, w) > 0 \text{ or } \pi(w, v) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Since U is finite, we may denote the connected components of H as $(U_i)_{i=1}^n$.

For any $v, w \in U$ such that $v \stackrel{H}{\sim} w$ (that is $\pi(v, w) > 0$ or $\pi(w, v) > 0$), the complementary slackness theorem gives $|\phi^*(v) - \phi^*(w)| = d(v, w)$. Therefore,

$$|\lfloor \phi^* \rfloor(v) - \lfloor \phi^* \rfloor(w)| = |\phi^*(v) - \delta_v - \phi^*(w) + \delta_w| = |\pm d(v, w) - (\delta_v - \delta_w)|,$$

which implies that $\delta_v - \delta_w$ has an integer value. Since $\delta_v - \delta_w \in (-1, 1)$, it must be 0. In conclusion, $\delta_v = \delta_w$ for all $v \stackrel{H}{\sim} w$. By transitivity, δ_v is constant throughout all vertices v in the connected component U_i , for each i . We may then define $\delta_i := \delta_v$ for any $v \in U_i$. Now observe that

$$\sum_{v \in U_i} \mu_1(v) = \sum_{v \in U_i} \sum_{w \in \text{supp}(\mu_2)} \pi(v, w) = \sum_{v \in U_i} \sum_{w \in U_i} \pi(v, w),$$

because for each $v \in U_i$ and each $w \in \text{supp}(\mu_2)$ such that $\pi(v, w) > 0$, we know that $v \stackrel{H}{\sim} w$, so $w \in U_i$. By similar arguments,

$$\sum_{w \in U_i} \mu_2(w) = \sum_{w \in U_i} \sum_{v \in \text{supp}(\mu_1)} \pi(v, w) = \sum_{w \in U_i} \sum_{v \in U_i} \pi(v, w),$$

and therefore $\sum_{v \in U_i} \mu_1(v) = \sum_{v \in U_i} \mu_2(v)$ for all i . Intuitively, this equation means the conservation of mass within each connected component U_i , since no mass is transported by π into or out of U_i (by definition of H). Lastly, we can compute that

$$\begin{aligned} \sum_{v \in V} [\phi^*](v)(\mu_1(v) - \mu_2(v)) &= \sum_{v \in V} \phi^*(v)(\mu_1(v) - \mu_2(v)) - \sum_{v \in V} \delta_v(\mu_1(v) - \mu_2(v)) \\ &= W_1(\mu_1, \mu_2) - \sum_{v \in U} \delta_v(\mu_1(v) - \mu_2(v)) \\ &= W_1(\mu_1, \mu_2) - \sum_{i=1}^n \sum_{v \in U_i} \delta_v(\mu_1(v) - \mu_2(v)) \\ &= W_1(\mu_1, \mu_2) - \sum_{i=1}^n \delta_i \left(\sum_{v \in U_i} (\mu_1(v) - \mu_2(v)) \right) \\ &= W_1(\mu_1, \mu_2). \end{aligned}$$

Therefore $[\phi^*]$ is an optimal Kantorovich potential as desired. \square

3 The idleness function is 3-piece linear

In this section, we will prove one of the main results: for any $x, y \in V$ such that $d(x, y) \geq 2$, the idleness function $p \mapsto \kappa_p(x, y)$ is piecewise linear with at most 3 linear parts. The proof follows the method from Theorem 3.4 in [1], which proves the result in case $x \sim y$. First, we need the following lemma.

Lemma 3.1. *Let $G = (V, E)$ be a locally finite and connected graph, and let $x, y \in V$ with $d(x, y) = \delta \geq 2$. Given $p \in (0, 1]$, every optimal Kantorovich potential $\phi^* : V \rightarrow \mathbb{R}$ transporting μ_x^p to μ_y^p satisfies*

$$\delta - 2 \leq \phi^*(x) - \phi^*(y) \leq \delta.$$

Moreover, if $p > \frac{1}{2}$, then

$$\phi^*(x) - \phi^*(y) = \delta.$$

Proof. Let $\pi^* \in \Pi(\mu_x^p, \mu_y^p)$ and $\phi^* : V \rightarrow \mathbb{R}$ be an optimal plan and an optimal Kantorovich potential transporting μ_x^p to μ_y^p .

The marginal constraints of π^* give $\sum_{v \in B_1(x)} \pi^*(v, y) = \mu_y^p(y) = p > 0$, which implies that $\pi^*(x', y) > 0$ for some $x' \in B_1(x)$. By the complementary slackness theorem, $\phi^*(x') - \phi^*(y) = d(x', y)$. By the Lipschitz and metric properties, we then have

$$\delta \geq \phi^*(x) - \phi^*(y) \geq (\phi^*(x') - 1) - \phi^*(y) = d(x', y) - 1 \geq d(x, y) - 2 = \delta - 2.$$

Now, assume $p > \frac{1}{2}$. The marginal constraints of π^* give

$$\frac{1-p}{d_x} = \mu_x^p(v) = \sum_{w \in B_1(y)} \pi^*(v, w) \geq \pi^*(v, y) \quad \text{for all } v \in S_1(x),$$

and therefore

$$\sum_{v \in B_1(x)} \pi^*(v, y) = \mu_y^p(y) = p > 1 - p = \sum_{v \in S_1(x)} \mu_x^p(v) \geq \sum_{v \in S_1(x)} \pi^*(v, y).$$

Subtracting the rightmost term from the leftmost term gives $\pi^*(x, y) > 0$, and the complementary slackness theorem implies that

$$\phi^*(x) - \phi^*(y) = d(x, y) = \delta.$$

\square

Theorem 3.2. Let $G = (V, E)$ be a locally finite and connected graph, and let $x, y \in V$ with $d(x, y) = \delta \geq 2$. Then $p \mapsto \kappa_p(x, y)$ is concave and piecewise linear over $[0, 1]$ with at most 3 linear parts.

Proof. For $\phi : V \rightarrow \mathbb{R}$, define $F(\phi) := \frac{1}{d_x} \sum_{w \sim x} \phi(w) - \frac{1}{d_y} \sum_{w \sim y} \phi(w)$,

and for $j \in \{\delta - 2, \delta - 1, \delta\}$, define a set

$$A_j := \left\{ \phi : V \rightarrow \mathbb{Z} \mid \phi \in 1\text{-Lip}, \phi(x) = j, \phi(y) = 0 \right\}.$$

Moreover, define a constant $c_j := \sup_{\phi \in A_j} F(\phi)$. A linear function $f_j : \mathbb{R} \rightarrow \mathbb{R}$ is then defined by $f_j(t) := t \cdot j + (1 - t)c_j$. It follows that

$$\begin{aligned} W_1(\mu_x^p, \mu_y^p) &= \sup_{\phi \in 1\text{-Lip}} \sum_{w \in V} \phi(w)(\mu_x^p(w) - \mu_y^p(w)) \\ &\stackrel{\text{Lemma 3.1}}{=} \sup_{\substack{\phi \in 1\text{-Lip} \\ \phi(x) - \phi(y) \in [\delta - 2, \delta]}} \sum_{w \in V} \phi(w)(\mu_x^p(w) - \mu_y^p(w)) \\ &\stackrel{\text{Prop. 2.5}}{=} \sup_{\substack{\phi \in 1\text{-Lip} \\ \phi : V \rightarrow \mathbb{Z} \\ \phi(y) = 0 \\ \phi(x) \in \{\delta - 2, \delta - 1, \delta\}}} \sum_{w \in V} \phi(w)(\mu_x^p(w) - \mu_y^p(w)) \\ &= \sup_{\substack{\phi \in 1\text{-Lip} \\ \phi : V \rightarrow \mathbb{Z} \\ \phi(y) = 0 \\ \phi(x) \in \{\delta - 2, \delta - 1, \delta\}}} \left(p\phi(x) + \frac{1-p}{d_x} \sum_{w \sim x} \phi(w) - \frac{1-p}{d_y} \sum_{w \sim y} \phi(w) \right) \\ &= \max_{j \in \{\delta - 2, \delta - 1, \delta\}} \sup_{\phi \in A_j} (p \cdot j + (1 - p)F(\phi)) \\ &= \max_{j \in \{\delta - 2, \delta - 1, \delta\}} (p \cdot j + (1 - p)c_j) \\ &= \max\{f_{\delta-2}(p), f_{\delta-1}(p), f_{\delta}(p)\}, \end{aligned} \tag{3.1}$$

and therefore

$$\kappa_p(x, y) = 1 - \frac{1}{\delta} \max\{f_{\delta-2}(p), f_{\delta-1}(p), f_{\delta}(p)\}. \tag{3.2}$$

Hence, $p \mapsto \kappa_p(x, y)$ is concave and piecewise linear with at most 3 linear parts. \square

Remark 3.3. For $p > \frac{1}{2}$, in the second line of equations (3.1), the condition on the supremum can be replaced by $\phi(x) - \phi(y) = \delta$, due to the second half of Lemma 3.1. Doing so gives $W_1(\mu_x^p, \mu_y^p) = f_{\delta}(p)$ for all $p > \frac{1}{2}$. In other words, the idleness function $p \mapsto \kappa_p(x, y)$ has the last linear part (at least) on the interval $[\frac{1}{2}, 1]$. The same statement is also true in case $x \sim y$ (see [1], Theorem 4.4). One immediate consequence is the simplification of the Lin-Lu-Yau curvature.

Corollary 3.4. Let $x \neq y \in V$. The Lin-Lu-Yau curvature satisfies

$$\kappa_{LLY}(x, y) = \frac{\kappa_p(x, y)}{1 - p}$$

for all $p \in [\frac{1}{2}, 1)$.

4 Critical points of the idleness function

In this section, we will discuss about the length of each linear part of the idleness function in terms of “critical points”.

Definition 4.1 (critical points). Define *critical points* (of $\kappa_p(x, y)$) to be the values $p^* \in (0, 1)$ such that

$$\lim_{p \rightarrow p^*} \frac{\partial}{\partial p} \kappa_p(x, y) \neq \lim_{p \rightarrow p^*} \frac{\partial}{\partial p} \kappa_p(x, y).$$

In other words, critical points are the values of p where the changes of slopes of the function $p \mapsto \kappa_p(x, y)$ happen. We may replace $\kappa_p(x, y)$ by $W_1(\mu_x^p, \mu_y^p)$ because they are closely related by the linear relation:

$$\kappa_p(x, y) = 1 - \frac{W_1(\mu_x^p, \mu_y^p)}{d(x, y)},$$

so they share the same critical points. Since the idleness function has at most 3 linear pieces, there are at most two critical points. Our goal of this section is to determine the possible values of the critical points.

The following proposition shows that, in case of $d_x = 1$ or $d_y = 1$, the idleness function is actually linear on the entire interval $[0, 1]$, so there is no critical point. As a consequence, all results about critical points will be discussed under the assumption that $d_x, d_y \geq 2$.

Proposition 4.1. *Let $G = (V, E)$ be a locally finite and connected graph, and let $x, y \in V$ with $d(x, y) \geq 2$. If $d_x = 1$ or $d_y = 1$, then $p \mapsto \kappa_p(x, y)$ is a linear function.*

Proof. Without loss of generality, assume that $d_x = 1$ and let $x \sim x_1$. Observe that every geodesic starting from x must pass through x_1 . In other words,

$$d(x, w) = d(x, x_1) + d(x_1, w) = 1 + d(x_1, w) \quad (4.1)$$

holds true for all $w \in V$ such that $w \neq x$. In particular, it holds true for all $w \in B_1(y)$ because $x \notin B_1(y)$ as $d(x, y) \geq 2$.

Consider an optimal transport plan π from μ_x^p to μ_y^p . The distance $W_1(\mu_x^p, \mu_y^p)$ can be derived as

$$\begin{aligned} W_1(\mu_x^p, \mu_y^p) &= \sum_{w \in B_1(y)} \sum_{v \in B_1(x)} \pi(v, w) d(v, w) \\ &= \sum_{w \in B_1(y)} (\pi(x, w) d(x, w) + \pi(x_1, w) d(x_1, w)) \\ &\stackrel{(4.1)}{=} \sum_{w \in B_1(y)} (\pi(x, w) + (\pi(x, w) + \pi(x_1, w)) d(x_1, w)) \\ &= \mu_x^p(x) + \sum_{w \in B_1(y)} \mu_y^p(w) d(x_1, w), \end{aligned}$$

which is linear in p , so is $\kappa_p(x, y)$. Here, the second line of the equation above uses the fact that $B_1(x) = \{x, x_1\}$, and the last line uses the marginal constraints of π : $\sum_{w \in B_1(y)} \pi(x, w) = \mu_x^p(x)$ and $\pi(x, w) + \pi(x_1, w) = \sum_{v \in B_1(x)} \pi(v, w) = \mu_y^p(w)$. □

Next is the main theorem of this section, which gives an upper bound on the values of critical points. Such a bound is sharp, as shown and explained in the Section 5.

Theorem 4.2. *Let $G = (V, E)$ be a locally finite and connected graph, and let $x, y \in V$ with $d(x, y) = \delta \geq 2$ and $d_x, d_y \geq 2$. Let $p^* \in (0, 1)$ be a critical point of $\kappa_p(x, y)$. Then*

$$p^* \leq \frac{1}{2} - \frac{1}{2} \cdot \frac{d_x + d_y}{2d_x d_y - d_x - d_y}.$$

The key of the proof lies in the following two lemmas. The first one compares the terms c_j 's introduced in the proof of Theorem 3.2. The second one gives an explicit formula for critical points in terms of c_j 's.

Lemma 4.3. *With the same setup as above,*

$$-1 < c_{\delta-2} - c_{\delta-1} \leq c_{\delta-1} - c_{\delta} \leq 1 - \frac{1}{d_x} - \frac{1}{d_y}. \quad (4.2)$$

The proof of Lemma 4.3 is postponed towards the end of this section.

Lemma 4.4. *With the same setup as above, define constants $p_1, p_2 \in \mathbb{R}$ to be*

$$p_1 := \frac{c_{\delta-2} - c_{\delta-1}}{c_{\delta-2} - c_{\delta-1} + 1}$$

$$p_2 := \frac{c_{\delta-1} - c_{\delta}}{c_{\delta-1} - c_{\delta} + 1}.$$

Then, for all $t \in \mathbb{R}$, the functions f_j as defined in Theorem 3.2 satisfy

$$\max\{f_{\delta-2}(t), f_{\delta-1}(t), f_{\delta}(t)\} = \begin{cases} f_{\delta-2}(t) & \text{if } t \leq p_1 \\ f_{\delta-1}(t) & \text{if } p_1 \leq t \leq p_2 \\ f_{\delta}(t) & \text{if } p_2 \leq t. \end{cases} \quad (4.3)$$

Moreover, it follows from $\kappa_p(x, y) = 1 - \frac{1}{\delta} \max\{f_{\delta-2}(p), f_{\delta-1}(p), f_{\delta}(p)\}$ that for each $i \in \{1, 2\}$, p_i is a critical point if and only if $p_i \in (0, 1)$.

Proof of Lemma 4.4. First, note that the denominators of p_1 and p_2 are positive real numbers, due to Lemma 4.3. Next, we show that $p_1 \leq p_2$. Consider the function $g : (-1, \infty) \rightarrow \mathbb{R}$ defined by

$$g(t) := \frac{t}{t+1},$$

which is an increasing function on t .

Note that $p_1 = g(c_{\delta-2} - c_{\delta-1})$ and $p_2 = g(c_{\delta-1} - c_{\delta})$. Hence, Lemma 4.3 implies

$$p_1 = g(c_{\delta-2} - c_{\delta-1}) \leq g(c_{\delta-1} - c_{\delta}) = p_2.$$

Next, we compare $f_{\delta-2}$ and $f_{\delta-1}$. From the definition $f_j(t) = t \cdot j + (1-t)c_j$, we have

$$\begin{aligned} f_{\delta-1}(t) \geq f_{\delta-2}(t) &\Leftrightarrow t(c_{\delta-2} - c_{\delta-1} + 1) \geq c_{\delta-2} - c_{\delta-1} \\ &\Leftrightarrow t \geq \frac{c_{\delta-2} - c_{\delta-1}}{c_{\delta-2} - c_{\delta-1} + 1} = p_1. \end{aligned}$$

Similarly, a comparison between $f_{\delta-1}$ and f_{δ} gives:

$$f_{\delta}(t) \geq f_{\delta-1}(t) \Leftrightarrow t \geq \frac{c_{\delta-1} - c_{\delta}}{c_{\delta-1} - c_{\delta} + 1} = p_2.$$

By the above comparisons, we can then conclude the equation

$$\max\{f_{\delta-2}(t), f_{\delta-1}(t), f_{\delta}(t)\} = \begin{cases} f_{\delta-2}(t) & \text{if } t \leq p_1 \\ f_{\delta-1}(t) & \text{if } p_1 \leq t \leq p_2 \\ f_{\delta}(t) & \text{if } p_2 \leq t \end{cases}$$

as desired. Moreover, Lemma 4.3 implies that

$$(\delta-2) - c_{\delta-2} < (\delta-1) - c_{\delta-1} < \delta - c_{\delta}.$$

Since $\frac{\partial}{\partial t} f_j = j - c_j$, it means that the slopes of $f_{\delta-2}, f_{\delta-1}$, and f_{δ} are all different:

$$\frac{\partial}{\partial t} f_{\delta-2} < \frac{\partial}{\partial t} f_{\delta-1} < \frac{\partial}{\partial t} f_{\delta}.$$

The second statement in the lemma immediately follows by renaming the variable t as p with a further restriction $p \in [0, 1]$. □

Proof of Theorem 4.2. Recall the function g defined in the proof of Lemma 4.4. The monotonicity of g together with Lemma 4.3 implies that

$$p_1 \leq p_2 = g(c_{\delta-1} - c_\delta) \leq g\left(1 - \frac{1}{d_x} - \frac{1}{d_y}\right) = \frac{1}{2} - \frac{1}{2} \cdot \frac{d_x + d_y}{2d_x d_y - d_x - d_y},$$

which concludes the proof of the theorem. \square

Now we come back to prove Lemma 4.3.

Proof of Lemma 4.3. First, we prove the rightmost inequality:

$$c_{\delta-1} - c_\delta \leq 1 - \frac{1}{d_x} - \frac{1}{d_y}.$$

Consider $\phi^* : V \rightarrow \mathbb{Z}$ such that $\phi^* \in A_{\delta-1}$ and $F(\phi^*) = \sup_{\phi \in A_{\delta-1}} F(\phi) =: c_{\delta-1}$. Such a maximizer exists because for $\phi : V \rightarrow \mathbb{Z}$, the function $F(\phi)$ only takes discrete values in \mathbb{Z}/l where $l = \text{lcd}(d_x, d_y)$. We will give a partial ordering to the set of vertices V by the following rule.

A vertex $b \in V$ is said to be a *child* of a vertex $a \in V$ if $a \sim b$ and $\phi^*(a) - \phi^*(b) = 1$. Moreover, for $a, b \in V$, we give an ordering $b \preceq a$ and say b is a *descendant* of a if there exists an $n \in \mathbb{N}$ and $b_0, b_1, b_2, \dots, b_n \in V$ such that $b_0 = b$, $b_n = a$, and b_i is a child of b_{i+1} for all $0 \leq i < n$. Equivalently, $b = b_0 \sim b_1 \sim b_2 \sim \dots \sim b_n = a$, and

$$d(a, b) \leq n = \phi^*(a) - \phi^*(b).$$

On the other hand, the Lipschitz property implies $\phi^*(a) - \phi^*(b) \leq d(a, b)$, so

$$\phi^*(a) - \phi^*(b) = d(a, b) \text{ for all } b \preceq a. \quad (4.4)$$

Here $b \preceq a$ means $b = a$ or $b \prec a$ by convention, and the equation (4.4) is obviously true in case $b = a$. Note that \preceq is a partial ordering on V : reflexivity and transitivity come from the definition, and antisymmetry can be seen from the equation (4.4).

In particular, since $\phi^*(x) - \phi^*(y) = \delta - 1 < d(x, y)$, it implies that $y \not\preceq x$.

Define a set of vertices $V_x \subseteq V$ by

$$V_x := \{w \in V \mid w \preceq x\}.$$

Now, define a function $\phi' : V \rightarrow \mathbb{Z}$ by

$$\phi'(w) := \begin{cases} \phi^*(w) + 1 & \text{if } w \in V_x \\ \phi^*(w) & \text{otherwise.} \end{cases}$$

We will now show that ϕ' is 1-Lipschitz. It is sufficient to show that $\phi'(w) - \phi'(z) \leq 1$ for any $w, z \in V$ such that $w \sim z$. By definition of ϕ' , we have

$$\begin{aligned} \phi'(w) - \phi'(z) &= (\phi^*(w) + \mathbb{1}_{V_x}(w)) - (\phi^*(z) + \mathbb{1}_{V_x}(z)) \\ &= \phi^*(w) - \phi^*(z) + \mathbb{1}_{V_x}(w) - \mathbb{1}_{V_x}(z) \\ &\leq 1 + \mathbb{1}_{V_x}(w) - \mathbb{1}_{V_x}(z) \end{aligned}$$

which is less than or equal to 1, except when $\phi^*(w) - \phi^*(z) = 1$ and $\mathbb{1}_{V_x}(w) = 1$ and $\mathbb{1}_{V_x}(z) = 0$, simultaneously. These exception conditions would imply that z is a child of w , and $w \preceq x$, and $z \not\preceq x$, which is impossible as it contradicts to the transitivity of partial ordering. Therefore, ϕ' is 1-Lipschitz as desired. Moreover, $\phi' \in A_\delta$ (because $\phi'(x) = \phi^*(x) + 1 = \delta$ and $\phi'(y) = \phi^*(y) = 0$ since $y \notin V_x$).

Comparison between ϕ^* and ϕ' gives

$$\begin{aligned}
 c_\delta = \sup_{\phi \in A_\delta} F(\phi) &\geq F(\phi') = \frac{1}{d_x} \sum_{w \sim x} \phi'(w) - \frac{1}{d_y} \sum_{w \sim y} \phi'(w) \\
 &= \frac{1}{d_x} \sum_{w \sim x} \phi^*(w) - \frac{1}{d_y} \sum_{w \sim y} \phi^*(w) + \frac{1}{d_x} \sum_{\substack{w \sim x \\ w \in V_x}} 1 - \frac{1}{d_y} \sum_{\substack{w \sim y \\ w \in V_x}} 1 \\
 &= F(\phi^*) + \frac{1}{d_x} |S_1(x) \cap V_x| - \frac{1}{d_y} |S_1(y) \cap V_x| \\
 &= c_{\delta-1} + \frac{1}{d_x} |S_1(x) \cap V_x| - \frac{1}{d_y} |S_1(y) \cap V_x|, \tag{4.5}
 \end{aligned}$$

where $S_1(x)$ and $S_1(y)$ are the sets of neighbours of x and of y , respectively.

A simple bound on (4.5) will give

$$c_\delta - c_{\delta-1} \geq \frac{1}{d_x} |S_1(x) \cap V_x| - \frac{1}{d_y} |S_1(y) \cap V_x| \geq \frac{1}{d_x}(0) - \frac{1}{d_y}(d_y) = -1.$$

However, this inequality can be improved by the following 3-case separation.

- Case 1: $S_1(x) \cap V_x \neq \emptyset$ and $S_1(y) \cap V_x \neq S_1(y)$.

Then

$$\begin{aligned}
 c_\delta - c_{\delta-1} &\geq \frac{1}{d_x} |S_1(x) \cap V_x| - \frac{1}{d_y} |S_1(y) \cap V_x| \\
 &\geq \frac{1}{d_x}(1) - \frac{1}{d_y}(d_y - 1) = -1 + \frac{1}{d_x} + \frac{1}{d_y}.
 \end{aligned}$$

- Case 2: $S_1(x) \cap V_x = \emptyset$.

It means that x has no child and hence no descendant, i.e. $V_x = \{x\}$. Thus

$$c_\delta - c_{\delta-1} \geq \frac{1}{d_x} |S_1(x) \cap V_x| - \frac{1}{d_y} |S_1(y) \cap V_x| = 0.$$

- Case 3: $S_1(y) \cap V_x = S_1(y)$.

It means that $y' \prec x$ for all neighbours y' of y . We now define a new function $\phi'' : V \rightarrow \mathbb{Z}$ by

$$\phi''(w) := \begin{cases} \phi^*(w) + 1 & \text{if } w \neq y \\ \phi^*(w) & \text{if } w = y, \end{cases}$$

which is 1-Lipschitz and in A_δ (similar as to how ϕ' is 1-Lipschitz and in A_δ). It follows that

$$\begin{aligned}
 c_\delta = \sup_{\phi \in A_\delta} F(\phi) &\geq F(\phi'') = \frac{1}{d_x} \sum_{w \sim x} \phi''(w) - \frac{1}{d_y} \sum_{w \sim y} \phi''(w) \\
 &= \frac{1}{d_x} \sum_{w \sim x} (\phi^*(w) + 1) - \frac{1}{d_y} \sum_{w \sim y} (\phi^*(w) + 1) \\
 &= F(\phi^*) = c_{\delta-1}.
 \end{aligned}$$

Hence, $c_\delta - c_{\delta-1} \geq 0$.

From the three cases above, we can conclude the rightmost inequality in (4.2):

$$c_{\delta-1} - c_\delta \leq 1 - \frac{1}{d_x} - \frac{1}{d_y}.$$

Next, we use a similar method as above to prove the leftmost inequality:

$$-1 < c_{\delta-2} - c_{\delta-1}.$$

Define a set of vertices $\tilde{V}_x \subseteq V$ by

$$\tilde{V}_x := \{w \in V \mid x \preceq w\}.$$

Then define a function $\tilde{\phi} : V \rightarrow \mathbb{Z}$ by

$$\tilde{\phi}(w) := \begin{cases} \phi^*(w) - 1 & \text{if } w \in \tilde{V}_x, \\ \phi^*(w) & \text{otherwise.} \end{cases}$$

By similar arguments, the function $\tilde{\phi}$ is also 1-Lipschitz, and it is in $A_{\delta-2}$ (because $\tilde{\phi}(x) = \phi^*(x) - 1 = \delta - 2$ and $\phi'(y) = \phi^*(y) = 0$ since $y \notin \tilde{V}_x$). Comparison between ϕ^* and $\tilde{\phi}$ gives

$$\begin{aligned} c_{\delta-2} &= \sup_{\phi \in A_{\delta-2}} F(\phi) \geq F(\tilde{\phi}) = \frac{1}{d_x} \sum_{w \sim x} \tilde{\phi}(w) - \frac{1}{d_y} \sum_{w \sim y} \tilde{\phi}(w) \\ &= \frac{1}{d_x} \sum_{w \sim x} \phi^*(w) - \frac{1}{d_y} \sum_{w \sim y} \phi^*(w) - \frac{1}{d_x} \sum_{\substack{w \sim x \\ w \in \tilde{V}_x}} 1 + \frac{1}{d_y} \sum_{\substack{w \sim y \\ w \in \tilde{V}_x}} 1 \\ &= F(\phi^*) - \frac{1}{d_x} |S_1(x) \cap \tilde{V}_x| + \frac{1}{d_y} |S_1(y) \cap \tilde{V}_x| \\ &= c_{\delta-1} - \frac{1}{d_x} |S_1(x) \cap \tilde{V}_x| + \frac{1}{d_y} |S_1(y) \cap \tilde{V}_x|. \end{aligned} \quad (4.6)$$

By considering a geodesic from x to y , namely $x = v_0 \sim v_1 \sim v_2 \sim \dots \sim v_\delta = y$, we have that $v_1 \in S_1(x)$ and

$$\phi^*(v_1) = \phi^*(v_1) - \phi^*(y) \leq d(v_1, y) = \delta - 1 = \phi^*(x),$$

which implies that $x \not\preceq v_1$, i.e., $v_1 \notin \tilde{V}_x$. Hence, $|S_1(x) \cap \tilde{V}_x| < d_x$, and (4.6) then gives

$$c_{\delta-2} > c_{\delta-1} - \frac{1}{d_x}(d_x) + \frac{1}{d_y}(0) = c_{\delta-1} - 1,$$

yielding the leftmost inequality in (4.2).

Lastly, we will prove the middle inequality in (4.2), or equivalently,

$$c_{\delta-2} + c_\delta \leq 2c_{\delta-1}.$$

Let $\phi_{\delta-2} \in A_{\delta-2}$ and $\phi_\delta \in A_\delta$ be two 1-Lipschitz functions such that $c_{\delta-2} = F(\phi_{\delta-2})$ and $c_\delta = F(\phi_\delta)$. Consider the function $\Phi := \frac{1}{2}(\phi_{\delta-2} + \phi_\delta)$. From the definition, we know that Φ is also 1-Lipschitz, and $\Phi(v) \in \mathbb{Z}/2$ for all $v \in V$, and $\Phi(x) = \delta - 1$ and $\Phi(y) = 0$. Therefore,

$$\frac{1}{2}(c_{\delta-2} + c_\delta) = \frac{F(\phi_{\delta-2}) + F(\phi_\delta)}{2} = F\left(\frac{\phi_{\delta-2} + \phi_\delta}{2}\right) = F(\Phi) \leq \sup_{\phi \in A_{\delta-1}[\mathbb{Z}/2]} F(\phi),$$

where $A_j[\mathbb{Z}/2] := \left\{ \phi : V \rightarrow \mathbb{Z}/2 \mid \phi \in 1\text{-Lip}, \phi(x) = j, \phi(y) = 0 \right\}$ is defined similarly to the previously defined $A_j = \left\{ \phi : V \rightarrow \mathbb{Z} \mid \phi \in 1\text{-Lip}, \phi(x) = j, \phi(y) = 0 \right\}$.

Lastly, we are left to show that

$$\sup_{\phi \in A_{\delta-1}[\mathbb{Z}/2]} F(\phi) = \sup_{\phi \in A_{\delta-1}} F(\phi) =: c_{\delta-1},$$

where the proof is very similar to the proof of Lemma 5.1 in [1].

The inequality $\sup_{\phi \in A_{\delta-1}[\mathbb{Z}/2]} F(\phi) \geq c_{\delta-1}$ is trivial since $A_{\delta-1} \subseteq A_{\delta-1}[\mathbb{Z}/2]$.

On the other hand, choose a function $\phi_{\delta-1} \in A_{\delta-1}[\mathbb{Z}/2]$ such that $F(\phi_{\delta-1}) = \sup_{\phi \in A_{\delta-1}[\mathbb{Z}/2]} F(\phi)$. Note that $\phi_{\delta-1}(v) = \frac{1}{2}(\lfloor \phi_{\delta-1}(v) \rfloor + \lceil \phi_{\delta-1}(v) \rceil)$ for all $v \in V$, and $\lfloor \phi_{\delta-1} \rfloor, \lceil \phi_{\delta-1} \rceil \in A_{\delta-1}$. Therefore,

$$\sup_{\phi \in A_{\delta-1}[\mathbb{Z}/2]} F(\phi) = F(\phi_{\delta-1}) = F\left(\frac{\lfloor \phi_{\delta-1} \rfloor + \lceil \phi_{\delta-1} \rceil}{2}\right) = \frac{F(\lfloor \phi_{\delta-1} \rfloor) + F(\lceil \phi_{\delta-1} \rceil)}{2} \leq c_{\delta-1}$$

as desired. \square

Remark 4.5 (Length of the first linear part). Note that each $c_j \in \mathbb{Z}/l$ where $l := \text{lcm}(d_x, d_y)$. Therefore, p_1 and p_2 must be in the form of $\frac{a}{q+1} = \frac{a}{a+l}$ for some $a \in \mathbb{Z}$. Hence, the least possible value for a positive critical point is $\frac{1}{1+l}$. In other words, first linear part of the function $p \mapsto \kappa_p(x, y)$ is at least the interval $[0, \frac{1}{1+\text{lcm}(d_x, d_y)}]$.

Remark 4.6 (Length of the last linear part). Theorem 4.2 says that the last linear part of the function $p \mapsto \kappa_p(x, y)$ is at least the interval

$$\left[\frac{1}{2} - \frac{1}{2} \cdot \frac{d_x + d_y}{2d_x d_y - d_x - d_y}, 1 \right].$$

In a special case that vertices x and y have the same degree $d_x = d_y = D \geq 2$, each critical point p^* of $\kappa_p(x, y)$ satisfies

$$p^* \leq \frac{1}{2} - \frac{1}{2} \cdot \frac{2D}{2D^2 - 2D} = \frac{D-2}{2D-2}. \quad (4.7)$$

Moreover, from the definition, $c_j \in \mathbb{Z}/D$, so p_1, p_2 must be in the form of $\frac{a}{D+a}$ for some integer $1 \leq a \leq D-2$.

The next section provides an example of a graph with $d_x = d_y = D$ where the inequality (4.7) holds with equality, that is, a critical point occurs exactly at $\frac{D-2}{2D-2}$.

5 An important family of examples

In this section we aim to construct a graph $G = (V, E)$ with points $x, y \in G$ such that $d(x, y) \geq 2$ and the idleness function $p \mapsto \kappa_p(x, y)$ has three linear pieces and has one critical point as large as the one mentioned in (4.7).

Let m, n, k be arbitrary natural numbers (including zero). Define vertices of G to be

$$V := \{x, y\} \cup \{x_0, x_1, y_0, y_1\} \cup \bigcup_{i=1}^m \{x'_i, y'_i, v_i, w_i\} \cup \bigcup_{i=1}^n \{x''_i, y''_i, z_i\} \cup \bigcup_{i=1}^k \{x'''_i, y'''_i\},$$

and define edges of G to be

$$\begin{aligned} E := & \left\{ \{x, x_0\}, \{x_0, y_0\}, \{y_0, y\} \right\} \cup \left\{ \{x, x_1\}, \{x_1, y_1\}, \{y_1, y\} \right\} \cup \\ & \bigcup_{i=1}^m \left\{ \{x, x'_i\}, \{x'_i, v_i\}, \{v_i, w_i\}, \{w_i, y'_i\}, \{y'_i, y\} \right\} \cup \\ & \bigcup_{i=1}^n \left\{ \{x, x''_i\}, \{x''_i, z_i\}, \{z_i, y''_i\}, \{y''_i, y\} \right\} \cup \\ & \bigcup_{i=1}^k \left\{ \{x, x'''_i\}, \{x'''_i, y'''_i\}, \{y'''_i, y\} \right\} \cup \\ & \bigcup_{i=1}^m \left\{ \{x_0, y'_i\}, \{x'_i, y_1\} \right\} \cup \bigcup_{i=1}^n \left\{ \{x_0, y''_i\}, \{x''_i, y_1\} \right\} \cup \bigcup_{i=1}^k \left\{ \{x_0, y'''_i\}, \{x'''_i, y_1\} \right\}. \end{aligned}$$

If m, n or k is zero, we simply remove the related vertices and edges.

The graph G is shown in Figure 1 in case $m = n = k = 1$ (but the indexes m, n, k are kept in the labelling for clarity).

In the constructed graph G , we have $d(x, y) = 3$ and $d_x = d_y = D = 2 + m + n + k$. Our goal is to show that the function $p \mapsto W_1(\mu_x^p, \mu_y^p)$ has its critical points at $\frac{m}{D+m}$ and $\frac{m+n}{D+m+n}$. In particular, if $k = 0$, then the larger critical point coincides with $\frac{D-2}{2D-2}$, the maximum possible value mentioned in Remark 4.6.

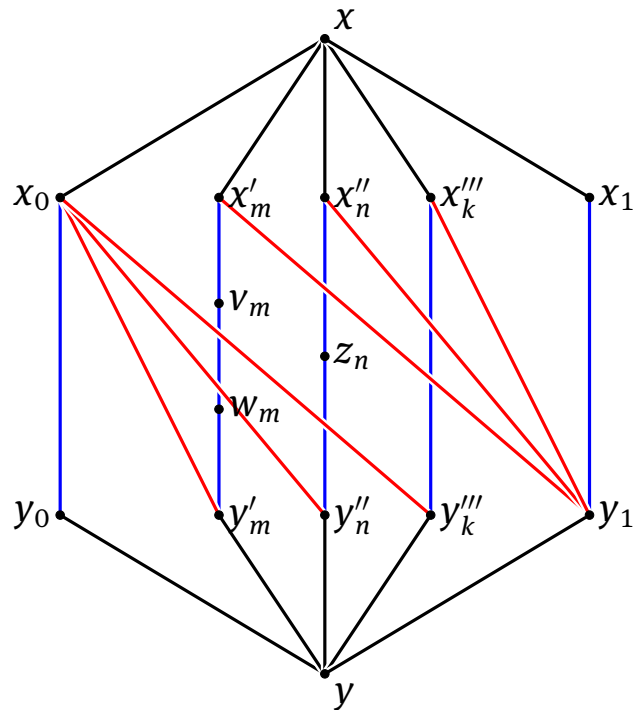


Figure 1: The constructed graph G with $m = n = k = 1$.

Recall the definitions, for $j \in \{1, 2, 3\}$,

$$c_j = \sup_{\phi \in A_j} F(\phi) = \sup_{\phi \in A_j} \frac{1}{D} \left(\sum_{w \sim x} \phi(w) - \sum_{w \sim y} \phi(w) \right),$$

and

$$f_j(p) = p \cdot j + (1 - p)c_j,$$

and

$$W_1(\mu_x^p, \mu_y^p) = \max\{f_1(p), f_2(p), f_3(p)\}.$$

We need to calculate the value of c_j for each $j \in \{1, 2, 3\}$. First, we start by giving a lower bound to c_j by choosing an appropriate function $\phi_j \in A_j$ for each j .

Table 1: Functions ϕ_j evaluated at the vertices of G .

$\phi_j(z)$	z														
	x	y	x_0	x_1	y_0	y_1	x'_i	v_i	w_i	y'_i	x''_i	z_i	y''_i	x'''_i	y'''_i
							$1 \leq i \leq m$				$1 \leq i \leq n$			$1 \leq i \leq k$	
ϕ_1	1	0	0	2	-1	1	2	1	0	-1	2	1	0	1	0
ϕ_2	2	0	1	2	0	1	2	1	0	0	2	1	0	1	0
ϕ_3	3	0	2	2	1	1	2	1	1	1	2	1	1	2	1

Define functions $\phi_1, \phi_2, \phi_3 : V \rightarrow \mathbb{Z}$ as in Table 1. It can be easily checked that $\phi_j \in A_j$ for $j \in \{1, 2, 3\}$, and hence we obtain the three following inequalities:

$$c_1 \geq F(\phi_1) = \frac{1}{D} \left((2m + 2n + k + 2) - (-m) \right) = \frac{3m + 2n + k + 2}{D} \quad (5.1)$$

$$c_2 \geq F(\phi_2) = \frac{1}{D} \left((2m + 2n + k + 3) - (1) \right) = \frac{2m + 2n + k + 2}{D} \quad (5.2)$$

$$c_3 \geq F(\phi_3) = \frac{1}{D} \left((2m + 2n + 2k + 4) - (m + n + k + 2) \right) = \frac{m + n + k + 2}{D} = 1. \quad (5.3)$$

Next, we give an upper bound to c_1 , c_2 , c_3 by calculating the costs of transport plans π_1, π_2, π_3 from μ_x^p to μ_y^p (with idleness $p = 0, \frac{m}{D+m}, \frac{m+n}{D+m+n}$, respectively). The plans π_1, π_2, π_3 are constructed as in Table 2.

Table 2: Transport plans π_j evaluated (non-vanishingly) at pairs of vertices of G .

$\pi_j(w, z)$			z					
			y	y_0	y_1	y'_i $1 \leq i \leq m$	y''_i $1 \leq i \leq n$	y'''_i $1 \leq i \leq k$
π_1	w	x						
		x_0		$\frac{1}{D}$				
		x_1			$\frac{1}{D}$			
		$x'_i (1 \leq i \leq m)$				$\frac{1}{D}$		
		$x''_i (1 \leq i \leq n)$					$\frac{1}{D}$	
		$x'''_i (1 \leq i \leq k)$						$\frac{1}{D}$
π_2	w	x				$\frac{1}{D+m}$		
		x_0		$\frac{1}{D+m}$				
		x_1			$\frac{1}{D+m}$			
		$x'_i (1 \leq i \leq m)$	$\frac{1}{D+m}$					
		$x''_i (1 \leq i \leq n)$					$\frac{1}{D+m}$	
		$x'''_i (1 \leq i \leq k)$						$\frac{1}{D+m}$
π_3	w	x				$\frac{1}{D+m+n}$	$\frac{1}{D+m+n}$	
		x_0		$\frac{1}{D+m+n}$				
		x_1			$\frac{1}{D+m+n}$			
		$x'_i (1 \leq i \leq m)$	$\frac{1}{D+m+n}$					
		$x''_i (1 \leq i \leq n)$	$\frac{1}{D+m+n}$					
		$x'''_i (1 \leq i \leq k)$						$\frac{1}{D+m+n}$

It is straightforward to check that $\pi_1 \in \Pi(\mu_x^0, \mu_y^0)$, $\pi_2 \in \Pi(\mu_x^{\frac{m}{D+m}}, \mu_y^{\frac{m}{D+m}})$, and $\pi_3 \in \Pi(\mu_x^{\frac{m+n}{D+m+n}}, \mu_y^{\frac{m+n}{D+m+n}})$, and therefore

$$\begin{aligned} c_1 = f_1(0) &\leq W_1(\mu_x^0, \mu_y^0) \leq \sum_{(w,z) \in V^2} \pi_1(w, z) d(w, z) \\ &= \frac{1}{D} (2 + 3m + 2n + k). \end{aligned} \quad (5.4)$$

Moreover,

$$\begin{aligned} \frac{2m}{D+m} + \frac{D}{D+m} c_2 &= f_2\left(\frac{m}{D+m}\right) \leq W_1(\mu_x^{\frac{m}{D+m}}, \mu_y^{\frac{m}{D+m}}) \\ &\leq \sum_{(w,z) \in V^2} \pi_2(w,z) d(w,z) \\ &= \frac{1}{D+m} (2 + 4m + 2n + k), \end{aligned}$$

which implies

$$c_2 \leq \frac{2m + 2n + k + 2}{D}. \quad (5.5)$$

Lastly,

$$\begin{aligned} \frac{3(m+n)}{D+m+n} + \frac{D}{D+m+n} c_3 &= f_3\left(\frac{m+n}{D+m+n}\right) \leq W_1(\mu_x^{\frac{m+n}{D+m+n}}, \mu_y^{\frac{m+n}{D+m+n}}) \\ &\leq \sum_{(w,z) \in V^2} \pi_3(w,z) d(w,z) \\ &= \frac{1}{D+m+n} (2 + 4m + 4n + k), \end{aligned}$$

which implies

$$c_3 \leq \frac{m+n+k+2}{D} = 1. \quad (5.6)$$

By comparing (5.1), (5.2), (5.3) to (5.4), (5.5), (5.6), we know that the exact values of c_j 's are

$$c_1 = \frac{3m + 2n + k + 2}{D}; \quad c_2 = \frac{2m + 2n + k + 2}{D}; \quad c_3 = 1.$$

Lemma 4.4 then gives a formula for $W_1(\mu_x^p, \mu_y^p)$ that

$$W_1(\mu_x^p, \mu_y^p) = \begin{cases} f_1(p) & \text{if } 0 \leq p \leq p_1 \\ f_2(p) & \text{if } p_1 \leq p \leq p_2 \\ f_3(p) & \text{if } p_2 \leq p \leq 1, \end{cases}$$

where the critical points are

$$p_1 = \frac{c_1 - c_2}{c_1 - c_2 + 1} = \frac{m}{D+m} \quad \text{and} \quad p_2 = \frac{c_2 - c_3}{c_2 - c_3 + 1} = \frac{m+n}{D+m+n}$$

as desired.

6 The Cartesian product

In [6] the authors proved the following results on the curvature of Cartesian products of graphs:

Theorem 6.1 ([6]). *Let $G = (V_G, E_G)$ be a d_G -regular graph and $H = (V_H, E_H)$ be a d_H -regular graph. Let $x_1, x_2 \in V_G$ with $x_1 \sim x_2$ and $y \in V_H$. Then*

$$\begin{aligned} \kappa_{LLY}^{G \times H}((x_1, y), (x_2, y)) &= \frac{d_G}{d_G + d_H} \kappa^G(x_1, x_2), \\ \kappa_0^{G \times H}((x_1, y), (x_2, y)) &= \frac{d_G}{d_G + d_H} \kappa_0^G(x_1, x_2). \end{aligned}$$

We now extend this result to the long-scale curvature.

Theorem 6.2. Let $G = (V_G, E_G)$ be a D_G -regular connected graph and $H = (V_H, E_H)$ be a D_H -regular connected graph. Let $x_1, x_2 \in V_G$ and $y_1, y_2 \in V_H$. Then

$$\kappa_{LLY}^{G \times H}((x_1, y_1), (x_2, y_2)) = \frac{D_G d(x_1, x_2) \kappa_{LLY}^G(x_1, x_2) + D_H d(y_1, y_2) \kappa_{LLY}^H(y_1, y_2)}{(D_G + D_H)(d(x_1, x_2) + d(y_1, y_2))}.$$

Furthermore, for all $p \in [\frac{1}{2}, 1)$, we have

$$\kappa_p^{G \times H}((x_1, y_1), (x_2, y_2)) = \frac{D_G d(x_1, x_2) \kappa_p^G(x_1, x_2) + D_H d(y_1, y_2) \kappa_p^H(y_1, y_2)}{(D_G + D_H)(d(x_1, x_2) + d(y_1, y_2))}.$$

Here we use the notation D_G, D_H , instead of d_G, d_H for the vertex degree to distinguish it from the distance function $d(\cdot, \cdot)$. Moreover, we use convention $d(x_1, x_2) \kappa_{LLY}^G(x_1, x_2) = 0$ in case $x_1 = x_2$, and $d(y_1, y_2) \kappa_{LLY}^H(y_1, y_2) = 0$ in case $y_1 = y_2$. Before proving the theorem, we introduce a lemma stating that the sum of 1-Lipschitz functions on two different graphs is a 1-Lipschitz function on the Cartesian product graph.

Lemma 6.3. Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two locally finite and connected graphs. Suppose $\phi_G : V_G \rightarrow \mathbb{R}$ and $\phi_H : V_H \rightarrow \mathbb{R}$ are 1-Lipschitz functions on G and H , respectively. Then the function $\phi : V_{G \times H} \rightarrow \mathbb{R}$ defined by

$$\phi((w, z)) := \phi_G(w) + \phi_H(z) \text{ for all } w \in V_G, z \in V_H$$

is also 1-Lipschitz function on the Cartesian product $G \times H$.

Proof of Lemma 6.3. Let $w_1, w_2 \in V_G$ and $z_1, z_2 \in V_H$. By applying 1-Lipschitz properties of ϕ_G and ϕ_H , we obtain

$$\begin{aligned} \phi((w_1, z_1)) - \phi((w_2, z_2)) &= \phi_G(w_1) + \phi_H(z_1) - \phi_G(w_2) - \phi_H(z_2) \\ &\leq d(w_1, w_2) + d(z_1, z_2) \\ &= d((w_1, z_1), (w_2, z_2)) \end{aligned}$$

yielding the lemma. □

Proof of Theorem 6.2. For idleness $p \in [0, 1]$, define idleness $\lambda, \lambda' \in [0, 1]$ to be

$$\lambda = \frac{pD_G + D_H}{D_G + D_H} \text{ and } \lambda' = \frac{D_G + pD_H}{D_G + D_H}. \quad (6.1)$$

The proof includes the four following steps:

- (1) Show that $W_1(\mu_{(x_1, y_1)}^p, \mu_{(x_2, y_2)}^p) \geq W_1(\mu_{x_1}^\lambda, \mu_{x_2}^\lambda) + W_1(\mu_{y_1}^{\lambda'}, \mu_{y_2}^{\lambda'})$.
- (2) Show that

$$\begin{aligned} W_1(\mu_{(x_1, y_1)}^p, \mu_{(x_2, y_2)}^p) &\leq \lambda' W_1(\mu_{x_1}^{p/\lambda'}, \mu_{x_2}^{p/\lambda'}) + (1 - \lambda') d(x_1, x_2) + \\ &\quad \lambda W_1(\mu_{y_1}^{p/\lambda}, \mu_{y_2}^{p/\lambda}) + (1 - \lambda) d(y_1, y_2). \end{aligned}$$

- (3) Show that the lower bound and upper bound of $W_1(\mu_{(x_1, y_1)}^p, \mu_{(x_2, y_2)}^p)$ given in (1) and (2) coincides for large enough $p \in [0, 1]$. Hence, the inequality in Step (1) is indeed an equality for p large enough.
- (4) Derive the Lin-Lu-Yau curvature on the Cartesian product.

Step (1) By Kantorovich Duality,

$$W_1(\mu_{(x_1, y_1)}^p, \mu_{(x_2, y_2)}^p) = \sup_{\substack{\phi: V_{G \times H} \rightarrow \mathbb{R} \\ \phi \in 1\text{-Lip}}} \sum_{\substack{w \in V_G \\ z \in V_H}} \phi((w, z)) (\mu_{(x_1, y_1)}^p((w, z)) - \mu_{(x_2, y_2)}^p((w, z))).$$

For each idleness $p \in [0, 1]$, let Φ_G^p and Φ_H^p be optimal Kantorovich potentials transporting $\mu_{x_1}^p$ to $\mu_{x_2}^p$, and $\mu_{y_1}^p$ to $\mu_{y_2}^p$, respectively. By Lemma 6.3, the function $\Phi((w, z)) := \Phi_G^\lambda(w) + \Phi_H^{\lambda'}(z)$ is 1-Lipschitz on $G \times H$, so it follows that

$$W_1(\mu_{(x_1, y_1)}^p, \mu_{(x_2, y_2)}^p) \geq \sum_{(w, z)} \left(\Phi_G^\lambda(w) + \Phi_H^{\lambda'}(z) \right) \left(\mu_{(x_1, y_1)}^p((w, z)) - \mu_{(x_2, y_2)}^p((w, z)) \right). \quad (6.2)$$

The idea is to decompose a measure in Cartesian product into a sum of measures in its coordinates. Consider characteristic equations of $\mu_{x_1}^\lambda$ and $\mu_{y_1}^{\lambda'}$:

$$\mu_{x_1}^\lambda(w) = \lambda \mathbb{1}_{x_1}(w) + \frac{(1-\lambda)}{D_G} \mathbb{1}_{S_1(x_1)}(w),$$

or equivalently

$$\mathbb{1}_{S_1(x_1)}(w) = \frac{D_G}{1-\lambda} \left(\mu_{x_1}^\lambda(w) - \lambda \mathbb{1}_{x_1}(w) \right). \quad (6.3)$$

Similarly,

$$\mathbb{1}_{S_1(y_1)}(z) = \frac{D_H}{1-\lambda'} \left(\mu_{y_1}^{\lambda'}(z) - \lambda' \mathbb{1}_{y_1}(z) \right). \quad (6.4)$$

Substitute (6.3) and (6.4) into the characteristic equation of $\mu_{(x_1, y_1)}^p((w, z))$:

$$\begin{aligned} \mu_{(x_1, y_1)}^p((w, z)) &= p \mathbb{1}_{x_1}(w) \mathbb{1}_{y_1}(z) + \frac{1-p}{D_G + D_H} \left(\mathbb{1}_{S_1(x_1)}(w) \mathbb{1}_{y_1}(z) + \mathbb{1}_{x_1}(w) \mathbb{1}_{S_1(y_1)}(z) \right) \\ &= p \mathbb{1}_{x_1}(w) \mathbb{1}_{y_1}(z) + K \left(\mu_{x_1}^\lambda(w) - \lambda \mathbb{1}_{x_1}(w) \right) \mathbb{1}_{y_1}(z) + \\ &\quad K' \left(\mu_{y_1}^{\lambda'}(z) - \lambda' \mathbb{1}_{y_1}(z) \right) \mathbb{1}_{x_1}(w) \\ &= (p - K\lambda - K'\lambda') \mathbb{1}_{x_1}(w) \mathbb{1}_{y_1}(z) + K \mu_{x_1}^\lambda(w) \mathbb{1}_{y_1}(z) + K' \mu_{y_1}^{\lambda'}(z) \mathbb{1}_{x_1}(w) \end{aligned}$$

with constants $K = \frac{(1-p)D_G}{(1-\lambda)(D_G+D_H)}$ and $K' = \frac{(1-p)D_H}{(1-\lambda')(D_G+D_H)}$.

It follows that

$$\begin{aligned} \sum_{(w, z)} \Phi_G^\lambda(w) \mu_{(x_1, y_1)}^p((w, z)) &= (p - K\lambda - K'\lambda') \Phi_G^\lambda(x_1) + K \sum_w \Phi_G^\lambda(w) \mu_{x_1}^\lambda(w) + K' \Phi_G^\lambda(x_1) \\ &= (p - K\lambda - K'\lambda' + K') \Phi_G^\lambda(x_1) + K \sum_w \Phi_G^\lambda(w) \mu_{x_1}^\lambda(w). \end{aligned}$$

With particular choice of λ, λ' as in (6.1), we have $K = K' = 1$ and $p - K\lambda - K'\lambda' + K' = 0$. The equation above simply turns into

$$\sum_{(w, z)} \Phi_G^\lambda(w) \mu_{(x_1, y_1)}^p((w, z)) = \sum_w \Phi_G^\lambda(w) \mu_{x_1}^\lambda(w).$$

Similarly, the same equation holds when subindex 1 is replaced by 2 everywhere, and therefore

$$\begin{aligned} \sum_{(w, z)} \Phi_G^\lambda(w) \left(\mu_{(x_1, y_1)}^p((w, z)) - \mu_{(x_2, y_2)}^p((w, z)) \right) &= \sum_w \Phi_G^\lambda(w) \mu_{x_1}^\lambda(w) - \sum_w \Phi_G^\lambda(w) \mu_{x_2}^\lambda(w) \\ &= W_1(\mu_{x_1}^\lambda, \mu_{x_2}^\lambda). \end{aligned} \quad (6.5)$$

By similar arguments, we also have

$$\sum_{(w, z)} \Phi_H^{\lambda'}(z) \left(\mu_{(x_1, y_1)}^p((w, z)) - \mu_{(x_2, y_2)}^p((w, z)) \right) = W_1(\mu_{y_1}^{\lambda'}, \mu_{y_2}^{\lambda'}). \quad (6.6)$$

Combining (6.5) and (6.6) into (6.2), we obtain

$$W_1(\mu_{(x_1, y_1)}^p, \mu_{(x_2, y_2)}^p) \geq W_1(\mu_{x_1}^\lambda, \mu_{x_2}^\lambda) + W_1(\mu_{y_1}^{\lambda'}, \mu_{y_2}^{\lambda'}).$$

Step (2) By the metric property of W_1 ,

$$W_1(\mu_{(x_1, y_1)}^p, \mu_{(x_2, y_2)}^p) \leq W_1(\mu_{(x_1, y_1)}^p, \mu_{(x_2, y_1)}^p) + W_1(\mu_{(x_2, y_1)}^p, \mu_{(x_2, y_2)}^p). \quad (6.7)$$

For each $i \in \{1, 2\}$, the measure $\mu_{x_i}^{p/\lambda'}$ satisfies

$$\lambda' \cdot \mu_{x_i}^{p/\lambda'}(w) = \begin{cases} p & \text{if } w = x_i \\ \lambda' \left(\frac{1-p}{D_G} \right) = \frac{1-p}{D_G + D_H} & \text{if } w \sim x_i \\ 0 & \text{otherwise.} \end{cases}$$

Let $\pi_x \in \Pi(\mu_{x_1}^{p/\lambda'}, \mu_{x_2}^{p/\lambda'})$ be an optimal transport plan from $\mu_{x_1}^{p/\lambda'}$ to $\mu_{x_2}^{p/\lambda'}$. Consider a transport plan $\pi \in \Pi(\mu_{(x_1, y_1)}^p, \mu_{(x_2, y_1)}^p)$ given by

$$\begin{aligned} \pi((x_1, z), (x_2, z)) &= \frac{1-p}{D_G + D_H} \quad \text{for all } z \stackrel{H}{\sim} y \\ \pi((w_1, y_1), (w_2, y_1)) &= \lambda' \pi_x(w_1, w_2) \quad \text{for all } w_1, w_2 \in V_G \\ \pi((w_1, z_1), (w_2, z_2)) &= 0 \quad \text{everywhere else.} \end{aligned}$$

The cost of plan π then give an upper bound for $W_1(\mu_{(x_1, y_1)}^p, \mu_{(x_2, y_1)}^p)$:

$$\begin{aligned} W_1(\mu_{(x_1, y_1)}^p, \mu_{(x_2, y_1)}^p) &\leq \sum_{(w_1, z_1), (w_2, z_2)} \pi((w_1, z_1), (w_2, z_2)) d((w_1, z_1), (w_2, z_2)) \\ &= \frac{(1-p)D_H}{D_G + D_H} d(x_1, x_2) + \lambda' W_1(\mu_{x_1}^{p/\lambda'}, \mu_{x_2}^{p/\lambda'}) \\ &= (1-\lambda')d(x_1, x_2) + \lambda' W_1(\mu_{x_1}^{p/\lambda'}, \mu_{x_2}^{p/\lambda'}). \end{aligned} \quad (6.8)$$

By similar arguments, we can derive

$$W_1(\mu_{(x_2, y_1)}^p, \mu_{(x_2, y_2)}^p) \leq (1-\lambda)d(y_1, y_2) + \lambda W_1(\mu_{y_1}^{p/\lambda}, \mu_{y_2}^{p/\lambda}). \quad (6.9)$$

Combining (6.8) and (6.9) into (6.7) results in

$$\begin{aligned} W_1(\mu_{(x_1, y_1)}^p, \mu_{(x_2, y_2)}^p) &\leq \lambda' W_1(\mu_{x_1}^{p/\lambda'}, \mu_{x_2}^{p/\lambda'}) + (1-\lambda')d(x_1, x_2) + \\ &\quad \lambda W_1(\mu_{y_1}^{p/\lambda}, \mu_{y_2}^{p/\lambda}) + (1-\lambda)d(y_1, y_2). \end{aligned}$$

Step (3) The equation

$$W_1(\mu_{x_1}^\lambda, \mu_{x_2}^\lambda) = \lambda' W_1(\mu_{x_1}^{p/\lambda'}, \mu_{x_2}^{p/\lambda'}) + (1-\lambda')d(x_1, x_2) \quad (6.10)$$

does not hold true in general. However, we will show that it is true for all p large enough. Recall that

$$\lambda = \frac{pD_G + D_H}{D_G + D_H} \quad \text{and} \quad \frac{p}{\lambda'} = \frac{pD_G + pD_H}{D_G + pD_H}$$

which are both increasing functions of $p \in [0, 1]$ and reach value 1 when $p = 1$. We consider p large enough so that $\lambda, \frac{p}{\lambda'} \geq \frac{1}{2}$. Recall Lemma 4.4 that for all idleness $q \in [\frac{1}{2}, 1]$,

$$W_1(\mu_{x_1}^q, \mu_{x_2}^q) = f_\delta(q) := q \cdot \delta + (1-q)c_\delta$$

where $\delta = d(x_1, x_2)$. In other words,

$$\frac{W_1(\mu_{x_1}^q, \mu_{x_2}^q) - q \cdot \delta}{1-q} = c_\delta = \frac{W_1(\mu_{x_1}^r, \mu_{x_2}^r) - r \cdot \delta}{1-r}$$

for all idleness $q, r \in [\frac{1}{2}, 1]$.

Substitution $q = \lambda$ and $r = \frac{p}{\lambda'}$ gives

$$\begin{aligned} W_1(\mu_{x_1}^\lambda, \mu_{x_2}^\lambda) &= \lambda \cdot \delta + (1 - \lambda) \left(\frac{W_1(\mu_{x_1}^{p/\lambda'}, \mu_{x_2}^{p/\lambda'}) - \frac{p}{\lambda'} \cdot \delta}{1 - \frac{p}{\lambda'}} \right) \\ &= \lambda' W_1(\mu_{x_1}^{p/\lambda'}, \mu_{x_2}^{p/\lambda'}) + (1 - \lambda') \delta, \end{aligned}$$

where the second line uses the identity $1 - \lambda = \lambda' - p$, yielding the equation (6.10).

Similarly, for large p such that $\lambda', \frac{p}{\lambda} \geq \frac{1}{2}$, we also have

$$W_1(\mu_{y_1}^{\lambda'}, \mu_{y_2}^{\lambda'}) = \lambda W_1(\mu_{y_1}^{p/\lambda}, \mu_{y_2}^{p/\lambda}) + (1 - \lambda) d(y_1, y_2). \quad (6.11)$$

We can then conclude that, when $p \in [0, 1]$ is large enough (so that $p, \lambda, \lambda', \frac{p}{\lambda}, \frac{p}{\lambda'} \geq \frac{1}{2}$), the lower bound and the upper bound of $W_1(\mu_{(x_1, y_1)}^p, \mu_{(x_2, y_2)}^p)$ agree.

Step (4) The previous steps imply that, for large enough $p \in [0, 1]$,

$$W_1(\mu_{(x_1, y_1)}^p, \mu_{(x_2, y_2)}^p) = W_1(\mu_{x_1}^\lambda, \mu_{x_2}^\lambda) + W_1(\mu_{y_1}^{\lambda'}, \mu_{y_2}^{\lambda'}).$$

Finally, we can translate this relation in term of the Lin-Lu-Yau curvature, using Corollary 3.4: for any $a_1 \neq a_2 \in A$ and any $q \in [\frac{1}{2}, 1]$,

$$\kappa_{LLY}^A(a_1, a_2) \stackrel{\text{Cor 3.4}}{=} \frac{\kappa_q(a_1, a_2)}{1 - q} = \frac{1}{1 - q} \left(1 - \frac{W_1(\mu_{a_1}^q, \mu_{a_2}^q)}{d(a_1, a_2)} \right).$$

For abbreviation, $\kappa_{LLY}^{G \times H} = \kappa_{LLY}^{G \times H}((x_1, y_1), (x_2, y_2))$ and $\kappa_{LLY}^G = \kappa_{LLY}^G(x_1, x_2)$ and $\kappa_{LLY}^H = \kappa_{LLY}^H(y_1, y_2)$

$$\begin{aligned} \kappa_{LLY}^{G \times H} &= \frac{1}{1 - p} \left(1 - \frac{W_1(\mu_{x_1}^\lambda, \mu_{x_2}^\lambda) + W_1(\mu_{y_1}^{\lambda'}, \mu_{y_2}^{\lambda'})}{d(x_1, x_2) + d(y_1, y_2)} \right) \\ &= \frac{1}{1 - p} \left(1 - \frac{d(x_1, x_2) \cdot (1 - (1 - \lambda) \kappa_{LLY}^G) + d(y_1, y_2) \cdot (1 - (1 - \lambda') \kappa_{LLY}^H)}{d(x_1, x_2) + d(y_1, y_2)} \right) \\ &= \frac{d(x_1, x_2)(1 - \lambda) \kappa_{LLY}^G + d(y_1, y_2)(1 - \lambda') \kappa_{LLY}^H}{(1 - p)(d(x_1, x_2) + d(y_1, y_2))} \\ &= \frac{d(x_1, x_2) D_G \kappa_{LLY}^G + d(y_1, y_2) D_H \kappa_{LLY}^H}{(D_G + D_H)(d(x_1, x_2) + d(y_1, y_2))}. \end{aligned}$$

Furthermore, for all $p \in [\frac{1}{2}, 1)$, Corollary 3.4: $\kappa_p = (1 - p) \kappa_{LLY}$ allows us to replace κ_{LLY} by κ_p in the above equation, which completes the proof. \square

7 Long-scale behaviour

In most papers regarding Ollivier Ricci curvature, only the short-scale curvature is usually considered because the curvature given at an edge $x \sim y$ is a discrete analogue to the Ricci curvature given at a unit tangent vector. Moreover, a lower bound on the short-scale curvature implies the same lower bound for the curvature between any two points (see [11, Proposition 19]):

$$\text{If } \kappa_p(x, y) \geq \kappa \text{ for all } x \sim y, \text{ then } \kappa_p(x, y) \geq \kappa \text{ for all } x, y \in V.$$

However, restricting oneself only to the short-scale curvature could lead to some contradiction to the nature of particular graphs, e.g. the hexagonal tiling as illustrated in the following subsection. Later, we then discuss about some global implication of curvature signs.

7.1 The hexagonal tiling

Let $G = (V, E)$ be a graph of the hexagonal tiling (which may be either infinite tessellation, or finite tessellation, e.g. on a torus T^2).

Consider a pair of points (x, y) with distance $d(x, y) = 7$. There are 4 non-equivalent positions of y relative to x listed as y_1, y_2, y_3, y_4 as shown in Figure 2.

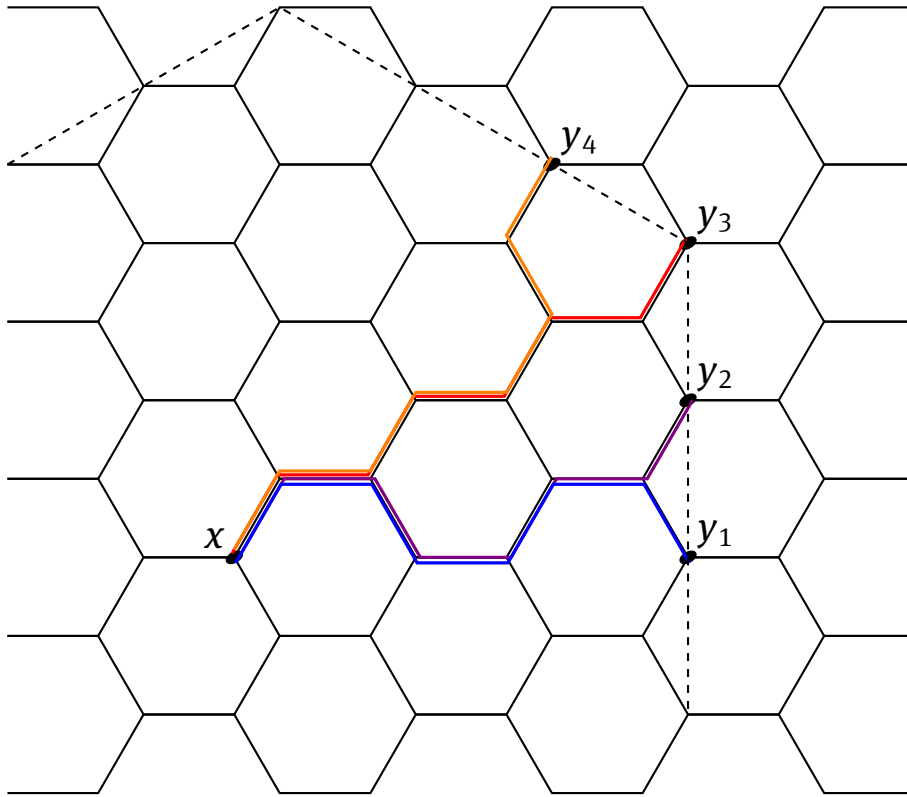


Figure 2: The dashed lines represent the locus of points with distance 7 from x in the hexagonal tiling.

The following proposition gives the formula of the short-scale and the long-scale curvature (of distance 7) in the hexagonal tiling.

Proposition 7.1. *Let $G = (V, E)$ be a graph of the hexagonal tiling. Let $p \in [0, 1]$ be an idleness parameter. Then*

(i) *for any $w, z \in V$ such that $w \sim z$, the curvature is given by*

$$\kappa_p(w, z) = -\frac{2}{3}(1 - p) < 0.$$

(ii) *for $x, y \in V$ such that $d(x, y) = 7$, the (long-scale) curvature is given by*

$$\kappa_p(x, y_1) = \kappa_p(x, y_2) = \frac{2}{21}(1 - p) > 0,$$

and

$$\kappa_p(x, y_3) = \kappa_p(x, y_4) = -\frac{2}{21}(1 - p) < 0.$$

Sketch of proof. (a) Given $w, z \in V$ with $w \sim z$. Denote the neighbors of w (other than z) by w_1, w_2 , and denote the neighbors of w (other than w) z_1, z_2 . An optimal way to transport μ_w^p to μ_z^p is to transport the mass p from w to z , and transport the masses $\frac{1-p}{3}$ from w_1 to w , from z to z_1 , and from w_2 to z_2 . Then

$$W_1(\mu_w^p, \mu_z^p) = p \cdot 1 + \frac{1-p}{3}(1 + 1 + 3) = 1 + \frac{2}{3}(1 - p),$$

which implies $\kappa_p(w, z) = 1 - W_1(\mu_w^p, \mu_z^p) = -\frac{2}{3}(1 - p)$.

(b) Given x, y with $d(x, y) = 7$ in the hexagon tiling. There are two possible configurations regarding the positions of the vertices in $S_1(x)$ and $S_1(y)$: either

(b1) two corresponding vertices in $S_1(x)$ and $S_1(y)$ have distance 5, and another pair has distance 9, or

(b2) two corresponding vertices in $S_1(x)$ and $S_1(y)$ have distance 9, and another pair has distance 5.

It can be checked that an optimal way to transport μ_x^p to μ_y^p is to transport the mass p from x to y , and transport the rest between the corresponding vertices mentioned above. This implies that

$$W_1(\mu_x^p, \mu_y^p) = \begin{cases} p \cdot 7 + \frac{1-p}{3}(5 + 5 + 9) = 7 - \frac{2}{3}(1-p) & \text{for case (b1)} \\ p \cdot 7 + \frac{1-p}{3}(9 + 9 + 5) = 7 + \frac{2}{3}(1-p) & \text{for case (b2),} \end{cases}$$

so $\kappa_p(x, y) = 1 - \frac{1}{7} W_1(\mu_x^p, \mu_y^p) = \pm \frac{2}{21}(1-p)$. The construction of these optimal transport plans are illustrated by the diagrams in Figure 3.

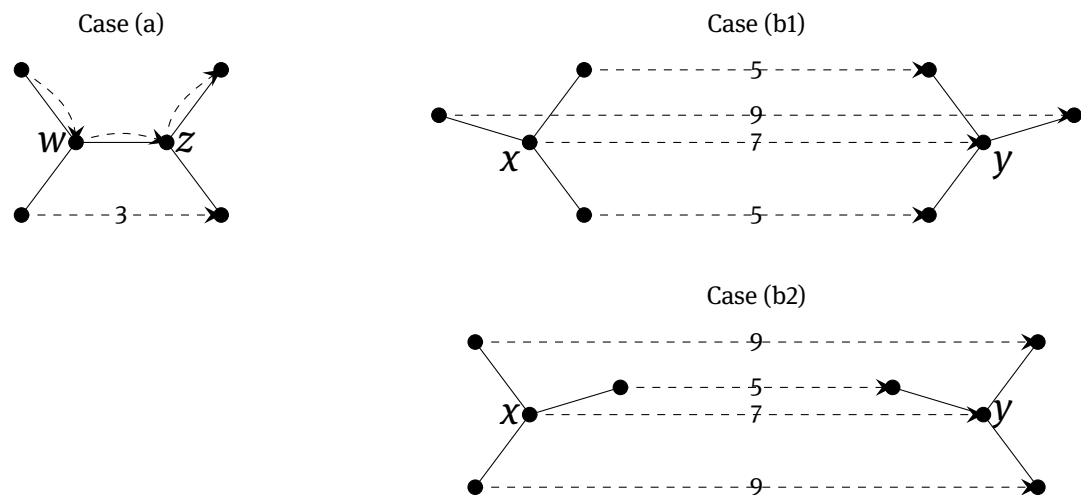


Figure 3: The diagrams show how to optimally transport from μ_w^p to μ_z^p in Case (a), and from μ_x^p to μ_y^p in Case (b1) and (b2). Masses are transported along the dashed arrows, without being split.

□

In particular, consider a finite hexagonal tessellation on a torus T^2 , where the space is expected to have both positive and negative curvature. However, the short-scale curvature is negative everywhere on the hexagonal tessellation, which suggests that the long-scale curvature is more suitable to describe this space.

7.2 Global results

Theorem 7.2 (non-positive curvature). *Let $G = (V, E)$ be a locally finite and connected graph and let $p \in [0, 1)$. Assume that the curvature $\kappa_p(x, y) \leq 0$ for all $x \neq y \in V$. Then G must be infinite.*

Proof. Suppose for the sake of contradiction that G is finite, with diameter

$$\text{diam}(G) := \sup\{d(w, z) : w, z \in V\} = L < \infty.$$

For a complete graph K_n , any edge $x \sim y$ satisfies $W_1(\mu_x^p, \mu_y^p) = |p - \frac{1-p}{n-1}|$, so $\kappa_p(x, y) > 0$ for every $p \in [0, 1)$. Since we assume G to be non-positively curved everywhere, G cannot be a complete graph, so $L \geq 2$.

Let x and y be antipodal vertices in V , that is $d(x, y) = L$. Consider a geodesic from x to y , namely $x = v_0 \sim v_1 \sim \dots \sim v_{L-1} \sim v_L = y$. It follows that v_1 is a neighbour of x , and v_{L-1} is a neighbour of y , and that $d(v_1, v_{L-1}) = L - 2$. Consider a transport plan $\pi \in \Pi(\mu_x^p, \mu_y^p)$ such that $\pi(v_1, v_{L-1}) > 0$.

Hence the $W_1(\mu_x^p, \mu_y^p)$ is bounded above by:

$$W_1(\mu_x^p, \mu_y^p) \leq \sum_{w,z \in V} \pi(w, z) d(w, z) \leq L \cdot \sum_{w,z \in V} \pi(w, z) = L. \quad (7.1)$$

Moreover, $\pi(w, z) d(w, z) < L \pi(w, z)$ when $w = v_1$ and $z = v_{L-1}$, so the inequality in (7.1) must be strict. That is $W_1(\mu_x^p, \mu_y^p) < L$ which then implies

$$\kappa_p(x, y) = 1 - \frac{1}{L} W_1(\mu_x^p, \mu_y^p) > 0,$$

contradicting to the curvature assumption. \square

Example 7.3. Let $G = (V, E)$ be the infinite D -regular tree (with $D \geq 2$). Given $p \in [0, 1]$ and let $x, y \in V$ such that $d(x, y) = L \geq 2$. Since there exists a unique geodesic between any two fixed vertices in the tree, an optimal transport plan from μ_x^p to μ_y^p happens in an obvious way. Thus we have

$$W_1(\mu_x^p, \mu_y^p) = p \cdot L + \frac{1-p}{D} (L - 2 + (D-1)(L+2)) = L + \frac{1-p}{D} (2D-4).$$

This implies

$$\kappa_p(x, y) = \frac{4-2D}{DL} (1-p).$$

In fact, this formula also holds in the case $d(x, y) = 1$. The infinite regular trees illustrate a family of graphs which have non-positive curvature everywhere.

Remark 7.4. The theorem of discrete Bonnet-Myers [6, 11] states that a graph $G = (V, E)$ with positive curvature bounded away from zero $\kappa_p(x, y) \geq K > 0$ for all $x \neq y \in V$ must be a finite graph. This assumption can be replaced by: $\kappa_p(x, y) \geq K > 0$ for all neighbours $x \sim y \in V$, since both assumptions are essentially equivalent. On the other hand, the assumption in Theorem 7.2 cannot be reduced to: $\kappa_p(x, y) \leq 0$ for all neighbours $x \sim y \in V$. As a counterexample, consider a graph G of a finite hexagonal tessellation on a torus T^2 (see Subsection 7.1).

There is another way to modify discrete Bonnet-Myers' theorem, by replacing the assumption condition with $\kappa_p(x, y) \geq \kappa > 0$ for a fixed vertex $x \in V$ and for all $y \in V \setminus \{x\}$.

Theorem 7.5 (modified discrete Bonnet-Myers). *Let $G = (V, E)$ be a locally finite and connected graph and let $p \in [0, 1]$. Assume that there is a constant $\kappa > 0$ and a fixed vertex $x \in V$ such that the curvature $\kappa_p(x, y) \geq \kappa$ for all $y \in V \setminus \{x\}$. Then G must be finite.*

The proof is very similar to the one in the original discrete Bonnet-Myers [11], which employs the Dirac-measure $\delta_x = \mathbb{1}_x$.

Proof. Consider any $y \in V \setminus \{x\}$. Let $L := d(x, y)$. The condition $\kappa_p(x, y) \geq \kappa$ implies that $W_1(\mu_x^p, \mu_y^p) \leq (1-\kappa)L$. Then

$$\begin{aligned} L &= W_1(\delta_x, \delta_y) \leq W_1(\delta_x, \mu_x^p) + W_1(\mu_x^p, \mu_y^p) + W_1(\mu_y^p, \delta_y) \\ &= W_1(\mu_x^p, \mu_y^p) + 2(1-p) \\ &\leq (1-\kappa)L + 2(1-p), \end{aligned}$$

which gives $d(x, y) \leq \frac{2(1-p)}{\kappa}$ for all $y \in V$. Hence, $d(y, z) \leq d(x, y) + d(x, z) \leq \frac{4(1-p)}{\kappa}$ for all $y, z \in V$. \square

Example 7.6. Let $G = (V, E)$ be the graph as shown in Figure 4. The short-scale Lin-Lu-Yau curvature of G can be computed as:

$$\kappa_{LLY}(x, w) = 1; \kappa_{LLY}(w, y) = -\frac{1}{3}; \kappa_{LLY}(y, z_1) = \kappa_{LLY}(y, z_2) = \frac{2}{3}.$$

Therefore G does not satisfy the condition of the original discrete Bonnet-Myers' theorem. However, the curvature between x and the other vertices can be computed as:

$$\kappa_{LLY}(x, w) = 1; \kappa_{LLY}(x, y) = \frac{1}{3}; \kappa_{LLY}(x, z_1) = \kappa_{LLY}(x, z_2) = \frac{2}{3}.$$

As a consequence, for any $p \in [\frac{1}{2}, 1)$, $\kappa_{LLY}(x, v) = \frac{\kappa_p(x, v)}{1-p} \geq \frac{1}{3(1-p)} > 0$ for all $v \in V \setminus \{x\}$, so G indeed satisfies the condition of Theorem 7.5.

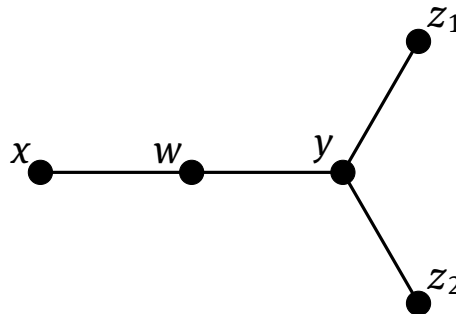


Figure 4: The graph G in Example 7.6.

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